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Phase diagram of the naive mean-field model for spin glasses

K Nishimura and K Nemoto

Institute of Physics, University of Tsukuba, Tsukuba 305, Japan

Abstract. The replica-symmetry-breaking solution is formulated for the infinite-ranged binomial spin glass model, which is a generalization of the Sherrington–Kirkpatrick (SK) model and includes the Bray–Sompolinsky–Yu (BSY) model as a limit. In the formulation it is found that the Parisi equation is the same as the original one for the SK model. This fact is used to show that the boundary between the spin glass phase and the mixed phase is independent of temperature, which is confirmed by numerical analysis of the naive mean-field equations and completes the phase diagram of the BSY model.

1. Introduction

The mean-field theory of spin glasses (SG) has been studied extensively on the basis of the Sherrington–Kirkpatrick (SK) model [1]. The SG phase of the SK model has been revealed to have stimulating properties such as the marginal stability, the ragged free energy, the hierarchically-structured metastable states and so on. These properties were derived originally from an interpretation of the replica-symmetry-breaking (RSB) solution for the SK model, and were thought to be rather specific to the model. More recent works, however, showed that this is not the case and that they are shared by other infinite-ranged SG models.

The Bray–Sompolinsky–Yu (BSY) model [2] is one of such models, which is described as the limit $m \to \infty$ of the infinite-ranged 'm-component binomial spin' Hamiltonian:

$$\mathcal{H} = -\frac{m}{2}\sum_{i,j} J_{ij}\hat{S}_i\hat{S}_j - mh\sum_i \hat{S}_i \tag{1.1}$$

where

$$\hat{S}_i = \frac{1}{m}\sum_{\mu=1}^{m} S_{i,\mu} \qquad (S_{i,\mu} = \pm 1) \tag{1.2}$$

are called 'binomial spins'. The interactions J_{ij} are usual Gaussian random variables with the mean J_0/N and the variance J^2/N, where N is the system size (we scale the unit of the energy to be $J = 1$ from now on). BSY [2] showed that in the limit $m \to \infty$ the equations of state are given by

$$m_i = \tanh\left(\beta\sum_j J_{ij}m_j + \beta h\right) \tag{1.3}$$

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where $\beta = 1/T$. These equations are called the naive mean-field (NMF) equations. The difference with those for the SK model, that is, the Thouless–Anderson–Palmer (TAP) equations, is the lack of the Onsager reaction terms in the mean-field.

BSY [2] solved the NMF equations by using Sompolinsky's method [3] and found the solution to be very similar to that for the SK model. Takayama and Nemoto [4] evaluated the number of solutions of (1.3) by using the replica method which was introduced for the TAP equations by Bray and Moore [5] and showed that it also increases exponentially with N in the SG phase. They also found numerically that the SG phase is marginally stable. More recently, we solved the NMF equations numerically to find that the number of solutions indeed agrees with the result from the replica method, and confirmed that the distribution of the overlap between metastable states is not trivial [6]. In this way, all the previous works indicate the existence of a great deal of similarity between the two models, and of the universality of the SG nature in the mean-field theory.

What is missing, however, is a complete and direct derivation of the RSB solution for the BSY model. The solution should be essentially equivalent to that obtained by Sompolinsky's method, as it is for the SK model [3,7]. To obtain the formulation of the RSB solution, we apply, in section 2, the RSB scheme proposed by De Dominicis *et al* (DGO) [8] to the binomial SG model, which includes both the SK model and the BSY model as mentioned above. It turns out that the formulation thus obtained is very similar to that for the original SK model. In particular, the Parisi equation holds for all m without any modification. By virtue of this fact, one can 'reuse' almost all arguments for the SK model to deduce the SG properties! One of the examples is the phase diagram in T–J_0 plane. We show that the boundary between the SG (RSB) phases with and without magnetization is parallel to the T-axis and is located at $J_0 = 1$ for the binomial model and thus for the BSY model. In section 3, we solve the NMF equations numerically to confirm that the boundary is indeed independent of T. Our results are summarized in section 4.

2. Replica-symmetry-breaking formulation

The straightforward application of the replica trick to the model Hamiltonian (1.1) gives the free energy as [2]

$$\mathcal{F} \equiv \frac{\beta F}{mnN} = \frac{\beta^2 m}{4n} \sum_{a,b} Q_{ab}^2 + \frac{\beta J_0}{2n} \sum_a M_a^2 - \frac{1}{mn} \ln Z_0 \tag{2.1}$$

where

$$Z_0 = \mathrm{Tr}_{(n)} \exp\left(\frac{1}{2}\sum_{a,b} Q_{ab}\sigma_a\sigma_b + \sum_a y_a\sigma_a\right) \tag{2.2}$$

$$\sigma_a = \beta \sum_{\mu=1}^{m} S_\mu^a = \beta m \bar{S}^a \qquad y_a = h + J_0 M_a . \tag{2.3}$$

In the above equations, a and b run from 1 to the number of replicas, n. By $\mathrm{Tr}_{(n)}$ we denote explicitly that the trace is taken over n-replicated binomial spins.

The auxiliary fields, or the order parameters, Q_{ab} and M_a, are determined self-consistently by

$$Q_{ab} = Q_{ba} = \langle \hat{S}^a \hat{S}^b \rangle \qquad M_a = \langle \hat{S}^a \rangle \tag{2.4}$$

where $\langle \cdots \rangle$ denotes the average generated by Z_0. The magnetization per Ising spin (component of the binomial spin) and the zero-field susceptibility are given by

$$M = \lim_{n\to 0} \frac{1}{n} \sum_a M_a \qquad \chi = \lim_{n\to 0} \frac{\beta m}{n} \sum_{ab} Q_{ab} \tag{2.5}$$

respectively.

We note here that the replica symmetry of M_a is expected not to be broken ($M_a = M$ for all a) since the mode associated with M is stable even in the RS solution [9].

2.1. *The Parisi matrix and DGO's RSB scheme*

To express the broken replica symmetry, Parisi introduced a hierarchical structure for the matrix $\mathbf{Q} = \{Q_{ab}\}$ (see [1]). As well known, the algebraic structure survives in the zero-replica limit as the order-parameter function $q(x)$ defined in $0 \leqslant x \leqslant 1$, and thus the free energy can be expressed as the functional of $q(x)$.

A set of the n dimensional Parisi matrices of level K is specified by a series of integers $n = p_0 > p_1 > \ldots > p_K = 1$ such that p_k divides p_{k-1}. Each element of the set, $\mathbf{A}$, is specified by $K+1$ parameters, $\{a_0, a_1, \ldots, a_K = \tilde{a}\}$, and is obtained as $\mathbf{A}_0$ in the series of $p_k \times p_k$ submatrices $\{\mathbf{A}_k\}$ defined recursively as

$$\mathbf{A}_k = \begin{pmatrix} \mathbf{A}_{k+1} & a_k \mathbf{U}_{k+1} & \cdots & a_k \mathbf{U}_{k+1} \\ a_k \mathbf{U}_{k+1} & \mathbf{A}_{k+1} & \cdots & a_k \mathbf{U}_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_k \mathbf{U}_{k+1} & a_k \mathbf{U}_{k+1} & \cdots & \mathbf{A}_{k+1} \end{pmatrix} \tag{2.6}$$

with $\mathbf{A}_K = \tilde{a}$, which represents the value of the diagonal elements. Here we denote by $\mathbf{U}_k$ the $p_k \times p_k$ matrix whose elements are all 1.

The replica symmetric solution is obtained by taking $\mathbf{Q}$ to be a Parisi matrix of level 1 with $q_0 = q$, $\tilde{q} = q + \tilde{\chi}/m$. For the SK model the RS solution is given by setting $m = 1$ and $\tilde{\chi} = 1 - q$, while $m \to \infty$ for the BSY model.

Although Parisi's RSB scheme works well for the binomial spin model with finite m, we adopt here another RSB scheme proposed by De Dominicis *et al* (DGO)[8]. The reason for this is that one must scale the anomalous response by m in each hierarchical block to attain the BSY limit ($m \to \infty$) properly, otherwise the limit goes to an unexpected replica-symmetric solution. By using DGO's RSB scheme, one can treat explicitly the anomalous response function $\Delta(x)$, which was first introduced by Sompolinsky [3].

In DGO's RSB scheme, the Parisi matrix $\mathbf{Q}$ is modified at the top level and takes the following structure:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_0 + \mathbf{D}_0 & \mathbf{Q}_0 & \cdots & \mathbf{Q}_0 \\ \mathbf{Q}_0 & \mathbf{Q}_0 + \mathbf{D}_0 & \cdots & \mathbf{Q}_0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Q}_0 & \mathbf{Q}_0 & \cdots & \mathbf{Q}_0 + \mathbf{D}_0 \end{pmatrix} \tag{2.7}$$

where $\mathbf{Q}_0$ and $\mathbf{D}_0$ are $p_0 \times p_0$ Parisi matrices described above, and p_0 is taken to divide n. The diagonal anomaly $\mathbf{D}_0$ is specified as

$$d_0 = \frac{\Delta_0}{mp_0} \qquad d_k - d_{k-1} = \frac{\Delta_k}{mp_k} \quad (1 \leqslant k \leqslant K-1) \qquad \tilde{d} = 0 \tag{2.8}$$

and the diagonal element of $\mathbf{Q}_0$ is given by

$$\tilde{q} = q_{K-1} + \frac{\tilde{\chi}_K}{m}. \tag{2.9}$$

To get a 'proper' solution, one should take the limit $p_0 \gg p_1 \gg \cdots \gg p_{K-1} \to \infty$ *before* taking $n \to 0$. After some calculation (see the Appendix for details), one gets the following expressions:

$$\mathcal{F} = \frac{\beta^2}{2}\left(\tilde{\chi}(1)q(1) + \int_0^1 q\dot{\Delta}\,\mathrm{d}x + \frac{\tilde{\chi}(1)^2}{2m}\right) + \frac{\beta J_0}{2}M^2 - \int \frac{\mathrm{d}z}{(2\pi q(0))^{1/2}} \exp\left(-\frac{z^2}{2q(0)}\right) G(0, z + h + J_0 M) \tag{2.10}$$

$$\dot{G}(x,y) + \tfrac{1}{2}\dot{q}G''(x,y) + \tfrac{1}{2}\dot{\Delta}G'(x,y)^2 = 0 \tag{2.11}$$

$$G(1,y) = \frac{1}{m}\mathrm{Tr}_{(1)} \exp\left(\frac{1}{2m}\tilde{\chi}(1)\sigma^2 + y\sigma\right) \tag{2.12}$$

where dots and primes denote the derivatives with respect to x and y, respectively. Note that the diffusion-like equation (the Parisi equation) (2.11) is the same as that for the SK model. The correspondent for Parisi's RSB scheme is given by choosing the Parisi gauge, $\dot{\Delta} = x\dot{q}$.

The main m dependence is in the initial condition for $G(1,y)$, (2.12). In fact, $G(1,y)$ reduces to that for the SK model by setting $m = 1$ and $\tilde{\chi}(1) = 1 - q(1)$, as expected:

$$G(1,y) = \tfrac{1}{2}\beta^2[1 - q(1)] + \ln(2\cosh\beta y). \tag{2.13}$$

On the other hand, the NMF limit $m \to \infty$ gives that for the BSY model:

$$G(1,y) = -\frac{\beta^2}{2}\tilde{\chi}(1)\widetilde{m}(1,y)^2 + \ln\{2\cosh\beta[y + \beta\tilde{\chi}(1)\widetilde{m}(1,y)]\} \tag{2.14}$$

$$\widetilde{m}(1,y) = \tanh\beta(y + \beta\tilde{\chi}(1)\widetilde{m}(1,y)). \tag{2.15}$$

In both cases, the Parisi equation (2.11) is common because it depends only on the structure of the RSB taken and the detail of the spin structure is not relevant.

The zero-field susceptibility (2.5) is given as

$$\chi = \beta\tilde{\chi}(0) \tag{2.16}$$

where

$$\tilde{\chi}(x) = \tilde{\chi}(1) + \int_x^1 \dot{\Delta}\,\mathrm{d}x \tag{2.17}$$

which means $\dot{\tilde{\chi}}(x) = -\dot{\Delta}$.

2.2. *Variational functional*

Sommers and Dupont [7] applied the Lagrange multiplier method for the SK model to obtain a closed form of equations of state. Their procedure can be naturally extended for the present model. In fact almost all equations obtained by them hold without any change.

The variational functional is given as the free energy (2.10) with two additional terms so that (2.11) and (2.12) hold:

$$\Phi = \mathcal{F} + \int \mathrm{d}y \; P(1,y)[G(1,y) - G_1(y)] - \int_0^1 \mathrm{d}x \int \mathrm{d}y P(x,y) \times \left(\dot{G}(x,y) + \tfrac{1}{2}\dot{q}G''(x,y) + \tfrac{1}{2}\dot{\Delta}G'(x,y)^2\right) \tag{2.18}$$

where $G_1(y)$ denotes the right-hand side (RHS) of (2.12). As for the SK model, the Lagrange multiplier $P(x,y)$ is introduced, which can be interpreted as the internal field distribution [7, 10]. The functional derivatives give (2.11), (2.12) and the following equations:

$$\dot{P} - \tfrac{1}{2}\dot{q}P'' + \beta\dot{\Delta}(P\widetilde{m})' = 0 \tag{2.19}$$

$$P(0,y) = (2\pi q(0))^{-1/2} \exp\left(-\frac{(y-h-J_0M)^2}{2q(0)}\right) \tag{2.20}$$

$$q(x) = \int \mathrm{d}y \; P(x,y)\widetilde{m}(x,y)^2 \tag{2.21}$$

$$M = \int \mathrm{d}y \; P(0,y)\widetilde{m}(0,y) \tag{2.22}$$

$$\beta\widetilde{\chi}(x) = \int \mathrm{d}y \; P(x,y)\widetilde{m}'(x,y) \tag{2.23}$$

where $\widetilde{m}(x,y) = TG'(x,y)$ is introduced, which satisfies

$$\dot{\widetilde{m}} + \tfrac{1}{2}\dot{q}\widetilde{m}'' + \beta\dot{\Delta}\widetilde{m}\widetilde{m}' = 0. \tag{2.24}$$

Many interesting properties of the above equations known for the SK model hold true for the present model. Among them, the most important is the following: The differentiation of (2.21) with respect to x gives the condition

$$\int \mathrm{d}y \; P(x,y)\widetilde{m}'(x,y)^2 = 1 \tag{2.25}$$

or $\dot{q} = 0$. This equation is equivalent to the condition of the marginal stability in the replica space. The phase boundary of the RSB (the de Almeida–Thouless line [9]) is given as the special case where $x = 1$.

Sompolinsky [3] and Sommers [11] argued that $q(0) = 0$ implies $\chi = 1$ for the SK model. Their argument holds true for the present case without modification. For $h = J_0 = 0$, $q(0) = 0$ is a solution of (2.21) since $\tilde{m}$ is an odd function of y so that $\tilde{m}(x, 0) = 0$. Then, (2.23) and (2.25) reduce to

$$\beta\tilde{\chi}(0) = \widetilde{m}'(0,0) \qquad \widetilde{m}'(0,0)^2 = 1 \tag{2.26}$$

respectively, which means $\chi = \beta\tilde{\chi}(0) = 1$. This result is also true for $J_0 \neq 0$ as long as $M = 0$, which is a solution of (2.22).

On the other hand, Toulouse [12] discussed the boundary of the spin glass phase and the mixed phase on J_0–T plane. If the equation of magnetization has the form $M = f(T, h + J_0 M)$, which is indeed satisfied in the present case, as seen in (2.22) with (2.20), then it can be expanded as

$$M = \chi(T, J_0 = 0)(h + J_0 M) + \text{non-linear terms.} \tag{2.27}$$

Thus the phase boundary $J_0^c(T)$ is given by

$$J_0^c(T) = \chi(T, J_0 = 0)^{-1}. \tag{2.28}$$

Therefore we can conclude that the phase boundary is parallel to the T-axis and at $J_0 = 1$ irrespective of the number of the internal Ising spins, m.

3. Numerical analysis

To confirm the validity of the prediction of the RSB solution we perform numerical analysis on the phase boundary between the SG and the mixed phase for the BSY model by means of the NMF equations (1.3). The procedure is as follows. We solve the NMF equations numerically for each sample from $T = 2.5$ down to 0.75 successively keeping J_0 fixed. This cooling procedure is repeated for several samples and for various J_0 between 0 and 1.5. The system sizes we examine are $N = 50$, 60, 80, 100, 120, 150, 200 and $200 \sim 50$ samples are used for random average. Solutions of (1.3) are given as local minima of the NMF free energy:

$$N f_{\mathrm{NMF}} = -\frac{1}{2}\sum_{i,j} J_{ij} m_i m_j - h\sum_i m_i + T\sum_i \left(\frac{1+m_i}{2}\ln\frac{1+m_i}{2} + \frac{1-m_i}{2}\ln\frac{1-m_i}{2}\right). \tag{3.1}$$

To find the local minimum we use relaxational dynamics described by the following equations:

$$\tau\frac{\mathrm{d}m_i}{\mathrm{d}t} = -N\frac{\partial f_{\mathrm{NMF}}}{\partial m_i} = \sum_i J_{ij} m_j + h - T\tanh^{-1} m_i. \tag{3.2}$$

A solution of (1.3) is given as a steady state of these equations. Using the solution obtained, we estimate the magnetization and the free energy to determine the boundary between the SG and mixed phases.

3.1. Magnetization

The magnetization of the system is evaluated as $M = N^{-1}\sum_i m_i$ for each sample. Because of the finite-size effect, magnetization appears for any J_0 and N (an infinitesimal field, $h \sim 10^{-4}$, is applied during the cooling process to break time-reversal symmetry). To extract the thermodynamic limit we fit these data to the following form,

$$M(N) = M(\infty) + aN^{-\phi} \tag{3.2}$$

where $M(\infty)$, a and ϕ are the fitting parameters. In figure 1 we show two typical results of the fit together with the raw data, one in the SG phase ($J_0 = 0.6$) and the other in the mixed phase ($J_0 = 1.2$). Although the fluctuation of raw data makes the error rather large, the extrapolated data clearly indicate which phase appears in the system. The J_0 dependence of the squared magnetization, M^2, is shown in figure 2 for several temperatures. For all temperatures investigated, the magnetization vanishes below J_0^c while $M^2 \sim J_0 - J_0^c$ above J_0^c with $J_0^c \sim 1$. This result indicates that J_0^c $(= 1)$ is independent of T and that the critical exponent of magnetization with respect to J_0 is $\frac{1}{2}$.

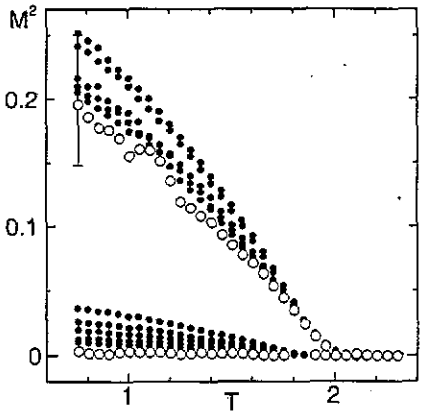

Figure 1. Temperature dependence of the magnetization at $J_0 = 1.2$ (upper) and at $J_0 = 0.6$ (lower) for $N = 50, 60, 100, 120$ and 200 (•) (top to bottom) and the thermodynamic limit (o).

3.2. Free energy

As easily seen from (3.1), the free energy is related to the magnetization through the equation

$$-\frac{\partial f_{\text{NMF}}}{\partial J_0} = \frac{1}{2}M^2 \tag{3.3}$$

which means the free energy is independent of J_0 in the paramagnetic and the SG phases. We show in figure 3 the free energy versus J_0 at $T = 0.8$. The free energy has a plateau region below $J_0^c \sim 1$ and then begins to decrease with increasing

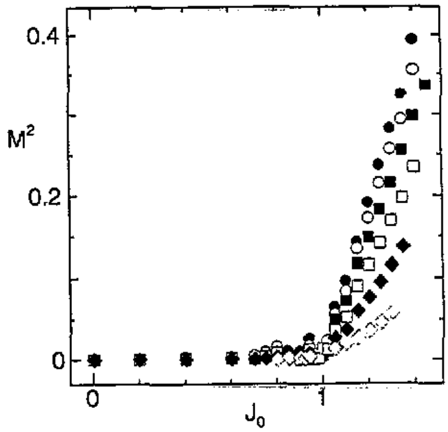

Figure 2. J_0 dependence of the squared magnetization at $T = 0.8(\bullet)$, $1.0(\circ)$, $1.2(\blacksquare)$, $1.4(\square)$, $1.6(\blacklozenge)$, $1.8(\lozenge)$.

J_0, as expected. To check the consistency of the data, we perform the numerical differentiation of the free energy to compare it with the previous result. As shown in the inset of figure 3 the results from direct observation and from the relation (3.3) are in perfect agreement quantitatively.

4. Summary

We have formulated the RSB solution for the binomial SG model, which interpolates between the SK model and the BSY model. The formulation turned out to be very similar to that of the original one for the SK model. In particular, the Parisi equation can be constructed so as to be independent of m. This similarity enables us to discuss the SG properties with the same arguments as used for the SK model. Thus we can conclude that the binomial SG model includes a class of the mean-field models in which the nature of the SG phase is common. In fact the TAP free energy for this model is easily evaluated by using Plefka's method [13], and is given by

$$f = -\frac{1}{2}\sum_{i,j} J_{ij} m_i m_j - h\sum_i m_i + \frac{1}{4m}\sum_{i,j} J_{ij}^2 (1 - m_i^2)(1 - m_j^2) + T\sum_i \left(\frac{1+m_i}{2} \ln \frac{1+m_i}{2} + \frac{1-m_i}{2} \ln \frac{1-m_i}{2} \right). \tag{4.1}$$

This is equivalent to that introduced in [4] with $\gamma = 1/m$. There it was emphasized that the Onsager term in the equations of state plays only a minor role in yielding the SG properties. The present result gives a complementary reasoning for them.

As an example of such common properties we have shown that the boundary between the SG and mixed phases is independent of temperature, that is, the boundary is vertical to the J_0-axis in T–J_0 plane. This verticality has been confirmed for the BSY model by numerical analysis of the NMF equations. This completes the phase diagram of the model in the T–J_0 plane as summarized in figure 4.

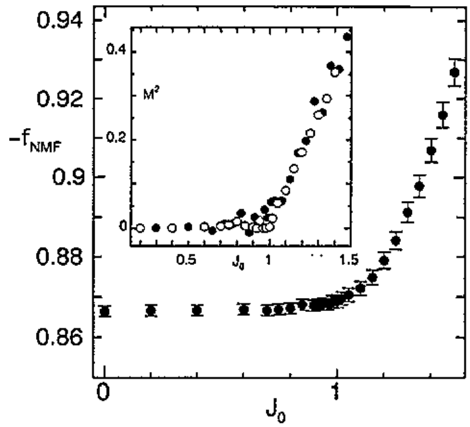

Figure 3. J_0 dependence of the free energy at $T = 1.0$. The inset shows the squared magnetization evaluated from numerical differentiation of the free energy(•) and that measured directly (o).

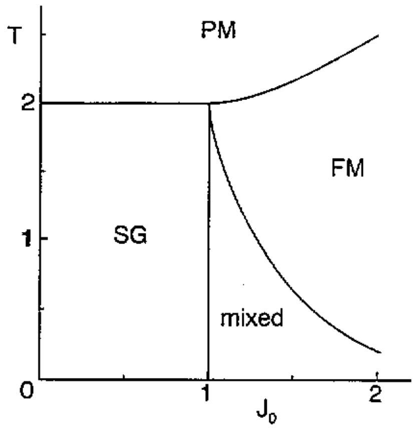

Figure 4. Phase diagram for the BSY model. PM, SG, FM and mixed represent the paramagnetic, spin glass, ferromagnetic and mixed phases. The PM–FM boundary is given as $T = J_0 + 1/J_0$.

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Appendix

It is convenient to write the 'partition function' given by (2.2) in matrix form as

$$Z_0(y) = \mathrm{Tr}_{(n)} \exp\left(\tfrac{1}{2}\sigma^T \mathbf{Q}\sigma + y u^T \sigma\right) \tag{A.1}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ are the n-dimensional column vectors whose elements are σ_a and 1, respectively. At the top level, $\mathbf{Q}$ is divided into $n/p_0 \times n/p_0$ blocks, so we can rewrite (A.1) as

$$Z_0(y) = \mathcal{Z}_0(\{y_i = y\}) \tag{A.2}$$

where

$$\mathcal{Z}_0(\{y_i\}) \equiv \mathrm{Tr}_{(n)} \exp\left[\frac{1}{2}\sum_i^{n/p_0} \boldsymbol{\sigma}_{0,i}^T \mathbf{D}_0 \boldsymbol{\sigma}_{0,i} + \frac{1}{2}\left(\sum_i^{n/p_0} \boldsymbol{\sigma}_{0,i}\right)^T \mathbf{Q}_0 \left(\sum_i^{n/p_0} \boldsymbol{\sigma}_{0,i}\right) + \sum_i^{n/p_0} y_i \boldsymbol{u}_0^T \boldsymbol{\sigma}_{0,i}\right] \tag{A.3}$$

or, more generally, we define

$$\mathcal{Z}_k(\{y_i\}) \equiv \mathrm{Tr}_{(np_k/p_0)} \exp\left[\frac{1}{2}\sum_i^{n/p_0} \boldsymbol{\sigma}_{k,i}^T (\mathbf{D}_k - d_{k-1}\mathbf{U}_k) \boldsymbol{\sigma}_{k,i} + \frac{1}{2}\left(\sum_i^{n/p_0} \boldsymbol{\sigma}_{k,i}\right)^T (\mathbf{Q}_k - q_{k-1}\mathbf{U}_k) \sum_i^{n/p_0} \boldsymbol{\sigma}_{k,i} + \sum_i^{n/p_0} y_i \boldsymbol{u}_k^T \boldsymbol{\sigma}_{k,i}\right]. \tag{A.4}$$

Here $\boldsymbol{\sigma}_{k,i}$ denotes the ith of the p_k dimensional column vectors and there are n/p_0 such vectors, so that there are a total of np_k/p_0 binomial spins (σ) in the exponent. Similarly $\boldsymbol{u}_k$ denotes the p_k dimensional column vector whose elements are all 1. Equation (A.4) reduces to (A.3) by setting $k = 0$, provided $q_{-1} = d_{-1} = 0$. The definition seems somewhat artificial but one will find it quite natural soon. Indeed it is a natural generalization of the recursive equation introduced by Duplantier [14].

Now we decompose the exponent into p_k/p_{k+1} blocks to lower the level by 1. Noting $\mathbf{U}_{k+1} = \boldsymbol{u}_{k+1}\boldsymbol{u}_{k+1}^T$ we have

$$\begin{aligned}
\mathcal{Z}_k(\{y_i\}) &= \mathrm{Tr}_{(np_k/p_0)} \exp\left\{\frac{1}{2}\sum_i^{n/p_0}\left[\sum_j^{p_k/p_{k+1}} \boldsymbol{\sigma}_{k+1,ij}^T (\mathbf{D}_{k+1} - d_k\mathbf{U}_{k+1}) \boldsymbol{\sigma}_{k+1,ij} + \delta d_k \left(\sum_j^{p_k/p_{k+1}} \boldsymbol{u}_{k+1}^T \boldsymbol{\sigma}_{k+1,ij}\right)^2\right] + \frac{1}{2}\left[\sum_j^{p_k/p_{k+1}}\left(\sum_i^{n/p_0} \boldsymbol{\sigma}_{k+1,ij}\right)^T\right.\right. \\
&\quad \left.\times (\mathbf{Q}_{k+1} - q_k\mathbf{U}_{k+1}) \sum_i^{n/p_0} \boldsymbol{\sigma}_{k+1,ij} + \delta q_k \left(\sum_i^{n/p_0}\sum_j^{p_k/p_{k+1}} \boldsymbol{u}_{k+1}^T \boldsymbol{\sigma}_{k+1,ij}\right)^2\right] \\
&\quad \left. + \sum_i^{n/p_0}\sum_j^{p_k/p_{k+1}} y_i \boldsymbol{u}_{k+1}^T \boldsymbol{\sigma}_{k+1,ij}\right\} \\
&= \int \frac{\mathrm{d}z}{(2\pi\delta q_k)^{1/2}} \exp\left(-\frac{z^2}{2\delta q_k}\right) \prod_i^{n/p_0} \int \frac{\mathrm{d}\eta_i}{(2\pi\delta d_k)^{1/2}} \exp\left(-\frac{\eta_i^2}{2\delta d_k}\right) \\
&\quad \times \mathrm{Tr}_{(np_k/p_0)} \exp\left\{\sum_j^{p_k/p_{k+1}}\left[\frac{1}{2}\sum_i^{n/p_0} \boldsymbol{\sigma}_{k+1,ij}^T (\mathbf{D}_{k+1} - d_k\mathbf{U}_{k+1}) \boldsymbol{\sigma}_{k,ij}\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\Bigg(\sum_i^{n/p_0}\sigma_{k+1,ij}\Bigg)^T(\mathbf{Q}_{k+1}-q_k\mathbf{U}_{k+1})\sum_i^{n/p_0}\sigma_{k+1,ij}\\
&+\sum_i^{n/p_0}(z+y_i+\eta_i)u_{k+1}^T\sigma_{k+1,ij}\Bigg]\Bigg\}\\
=&\int\frac{\mathrm{d}z}{(2\pi\delta q_k)^{1/2}}\exp\Big(-\frac{z^2}{2\delta q_k}\Big)\prod_i^{n/p_0}\int\frac{\mathrm{d}\eta_i}{(2\pi\delta d_k)^{1/2}}\exp\Big(-\frac{{\eta_i}^2}{2\delta d_k}\Big)\\
&\times\mathcal{Z}_{k+1}(\{z+y_i+\eta_i\})^{p_k/p_{k+1}}
\end{aligned}
\tag{A.5}
$$

where $\delta q_k = q_k - q_{k-1}$ and $\delta d_k = d_k - d_{k-1}$. To get the free energy per component Ising spin, we introduce

$$
\mathcal{G}_k(\{y_i\}) \equiv \frac{p_0}{mnp_k}\ln\mathcal{Z}_k(\{y_i\}) \qquad G_k(y)=\mathcal{G}_k(\{y_i=y\}). \tag{A.6}
$$

Then we get the following recursion formula:

$$
\begin{aligned}
G_k(y)=&\frac{p_0}{mnp_k}\ln\int\frac{\mathrm{d}z}{(2\pi\delta q_k)^{1/2}}\exp\Big(-\frac{z^2}{2\delta q_k}\Big)\prod_i^{n/p_0}\int\Big(\frac{mp_k}{2\pi\Delta_k}\Big)^{1/2}\mathrm{d}\eta_i\\
&\times\exp\Bigg\{mp_k\Bigg[-\frac{1}{2\Delta_k}\sum_i\eta_i^2+\frac{n}{p_0}\mathcal{G}_{k+1}(\{y+z+\eta_i\})\Bigg]\Bigg\}.
\end{aligned}
\tag{A.7}
$$

For $p_k \to \infty$ we can estimate the integration with respect to η_i by the saddle point method. Assuming that the saddle point $\{\eta_i^c\}$ is given at the symmetric point ($\eta_i^c = \eta_k^c$ for all i), we get

$$
\eta_k^c = \frac{n\Delta_k}{p_0}\left(\frac{\partial\mathcal{G}_{k+1}}{\partial\eta_i}\right)_{\{\eta_i=\eta_k^c\}} = \Delta_k\frac{\mathrm{d}G_{k+1}}{\mathrm{d}y}(z+y+\eta_k^c) \tag{A.8}
$$

and

$$
\begin{aligned}
G_k(y)\simeq&\frac{p_0}{mnp_k}\ln\int\frac{\mathrm{d}z}{(2\pi\delta q_k)^{1/2}}\exp\Big(-\frac{z^2}{2\delta q_k}\Big)\\
&\times\exp\Bigg\{\frac{mnp_k}{p_0}\Bigg[-\frac{1}{2\Delta_k}(\eta_k^c)^2+G_{k+1}(y+z+\eta_k^c)\Bigg]\Bigg\}\\
\simeq&\int\frac{\mathrm{d}z}{(2\pi\delta q_k)^{1/2}}\exp\Big(-\frac{z^2}{2\delta q_k}\Big)\\
&\times\Bigg[-\frac{1}{2\Delta_k}(\eta_k^c)^2+G_{k+1}(y+z+\eta_k^c)\Bigg]
\end{aligned}
\tag{A.9}
$$

where $mnp_k/p_0 \ll 1$ is used. This recursion terminates at $k = K$, where G_K is given by

$$
\begin{aligned}
G_K(y)=&\frac{p_0}{mn}\ln\mathrm{Tr}_{(n/p_0)}\exp\Bigg[\frac{1}{2}\sum_i^{n/p_0}(\tilde{\chi}_K/m-d_k)\sigma_i^2+y\sum_i^{n/p_0}\sigma_i\Bigg]\\
=&\frac{1}{m}\ln\mathrm{Tr}_{(1)}\exp\Big(\frac{1}{2m}\tilde{\chi}_K\sigma^2+y\sigma\Big).
\end{aligned}
\tag{A.10}
$$

Finally we can take the continuous limit where $K \to \infty$, $k/K \to x$, $\Delta_k \to \dot{\Delta}(x)\mathrm{d}x$, $\delta q_k \to \dot{q}(x)\mathrm{d}x$, $\eta_k^c(y) \to \dot{\Delta}(x)G'(x,y)\mathrm{d}x$ and $G_k(y) \to G(x,y)$:

$$\dot{G}(x,y) + \tfrac{1}{2}(\dot{q}(x)G''(x,y) + \dot{\Delta}(x)G'(x,y)^2) = 0 \qquad \text{(A.11)}$$

and

$$G(1,y) = \frac{1}{m} \ln \mathrm{Tr}_{(1)} \exp\left(\frac{1}{2m}\tilde{\chi}(1)\sigma^2 + y\sigma\right). \qquad \text{(A.12)}$$

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